Riemann zeros, prime numbers, and fractal potentials

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Using two distinct inversion techniques, the local one-dimensional potentials for the Riemann zeros and prime number sequence are reconstructed. We establish that both inversion techniques, when applied to the same set of levels, lead to the same fractal potential. This provides numerical evidence that the potential obtained by inversion of a set of energy levels is unique in one dimension. We also investigate the fractal properties of the reconstructed potentials and estimate the fractal dimensions to be $D=1.5$ for the Riemann zeros and $D=1.8$ for the prime numbers. This result is somewhat surprising since the nearest-neighbor spacings of the Riemann zeros are known to be chaotically distributed, whereas the primes obey almost Poissonlike statistics. Our findings show that the fractal dimension is dependent on both level statistics and spectral rigidity, Δ_3, of the energy levels.

I. INTRODUCTION

In the early work of Wu *et al.* [1] a general numerical algorithm was developed that could reconstruct a one-dimensional (1D) local potential from an essentially arbitrary set of levels (i.e., bound states). The basic idea behind the method (see Sec. II below) is to functionally minimize the potential so that it exactly reproduces the prescribed energy eigenvalues. This is, in principle, a solution of the 1D quantum inverse problem if one could fit the infinitely many levels to a local potential. Of course, this is beyond any numerical algorithm's ability, and one has to settle for fitting only the first N eigenvalues. Since the algorithm requires a direct solution of the Schrödinger equation (SE) for every eigenvalue, it is practically limited to fitting roughly $N \sim O(10^3)$ eigenvalues. In an interesting application of this algorithm, Wu and Sprung [2] chose as their eigenvalue spectrum the complex zeros (it is implicit throughout this paper that zeros refer to the complex zeros of the ζ function) of the Riemann zeta function, which is well known in number theory [3]. By finding a local 1D potential that exactly reproduced the first $N=500$ ''eigenvalues,'' they had effectively found a 1D quantum Hamiltonian for the first 500 zeros of the Riemann ζ function. This is, of course, interesting in its own right because such Hamiltonians have long been conjectured to hold the key to proving the celebrated Riemann hypothesis [3]. Their reconstructed potential was found to be fractal, and the fractal dimension was estimated to be $D=1.5$. This result suggests the intriguing possibility that an integrable system with time-reversal symmetry may generate the Riemann zeros provided the potential is fractal. It is worth commenting that the precise meaning one can attach to such 1D potentials is not clear. In particular, taking a chaotic system (i.e., the Riemann zeros) and forcing it to be integrable will lead to a potential that is dependent on the dimension of the Hilbert space (i.e., number of levels fitted).

Motivated by the work mentioned above, Ramani *et al.* [4] also investigated the reconstruction of a 1D system's

quantum Hamiltonian from its eigenvalues. Their approach to solving the inverse problem was to use a ''dressing transformation'' based on techniques developed in the solution of the nonlinear Korteweg–de Vries equation. Their method, being much more efficient numerically than the direct SE approach resulted in many more levels being fitted for the same numerical effort. Unfortunately, rather than directly examining the Riemann zeros, as Wu and Sprung had previously done, they investigated the fractal properties of potentials generated from energy levels whose nearest-neighbor spacing distribution (NNSD) were the Gaussian unitary ensemble (GUE) [5]. Although the NNSD of the Riemann zeros are also known to belong to the GUE, Ramani *et al.* could only make general comments about 1D potentials generated from the GUE statistics, but nothing specific about the Riemann zeros. Nevertheless, their work suggests, using $O(10^4)$ levels, a fractal dimension of $D=2$ as $N \rightarrow \infty$ for GUE reconstructed potentials. By extension, they conjectured that their more accurate estimate of the fractal dimension leads to a value of $D=2$ for the Riemann potential.

In a reply to this work, Wu and Sprung [6] have pointed out that using a different spectrum (i.e., a set of energy levels whose NNSD is the GUE rather than the actual complex Riemann zeros) does not necessarily imply that the fractal dimension of the reconstructed potential should be the same. One of the key reasons behind this statement is the lack of long-range correlations (i.e., the so-called spectral rigidity Δ_3) in the generic GUE spectrum used in Ref. [4]. Indeed, it was argued that if the fitting spectrum does not contain proper long-range correlations, the resulting potential may appear to be more random than the Riemann zeros potential. This, of course, can result in two different estimates for the fractal dimension even if a large number of levels have been fitted. Another important concern raised in their reply is that the potential obtained by inversion of the levels need not be unique. In other words, even if Ramani *et al.* had used the Riemann zeros as their fitting spectrum, they may not have obtained the same potential; that is, the fractal dimension

may depend on the inversion technique applied.

In light of the above, we believe that there are several questions that have been left unanswered. (i) For the same set of energy levels, are the potentials reconstructed from different inversion procedures unique? (ii) Does the fractal dimension depend on the inversion technique? (iii) What is the fractal dimension for the Riemann zeros potential? (iv) Is the fractal dimension a generic quantity for reconstructed 1D potentials or does it depend on the NNSD and Δ_3 statistics of the energy level spectrum?

In what follows, we will attempt to provide answers to these questions. We will begin by presenting a brief overview of the numerical inversion techniques used in Refs. [2,4], and then apply them to reconstruct the potentials of the Riemann zeros and prime numbers. Our decision to focus on these number sequences is threefold. First, the contradictory results $D=1.5$ and $D=2$ for the fractal dimension of the Riemann potential remain unresolved. Second, the Riemann zeros are known to be dual to the prime numbers in the sense that all of the primes are encoded through the complex Riemann zeros via Riemann's formula [3,7]. One may then speculate as to whether any additional relationships between the Riemann zeros and prime numbers can be obtained from the reconstructed fractal potentials. Finally, since the NNSD of the Riemann zeros belong to the GUE class, and the prime numbers obey a more poissonlike distribution, it is interesting to investigate the dependence of the fractal dimension on the level statistics. The reconstructed potentials obtained from the two inversion methods will then be used to address the questions of uniqueness and fractal dimension.

II. RECONSTRUCTING THE POTENTIAL FROM A SET OF LEVELS

In this section, we will give a brief overview of the two inversion techniques used in this paper to reconstruct the local 1D potentials. After this presentation, we will compare the reconstructed potentials for the Riemann zeros and prime number sequence obtained from both methods.

A. Direct SE approach

Perhaps the most intuitive approach to solving the inverse problem is to directly invoke the SE. One initially starts from some arbitrary potential $V(x)$ (although for reasons of convergence, it is wiser to start from the semiclassical potential as described in Ref. [2]) and solve the SE to obtain a set of eigenvalues ε_n. One then defines a "cost function"

$$F = \sum_n (\varepsilon_n - E_n)^2, \qquad (1)$$

where E_n are the exact eigenvalues one wishes to reproduce. Potential $V(x)$ is now adjusted in such a way as to minimize the cost function. Mathematically, this results in the following functional equation:

$$\frac{\delta F}{\delta V(x)} = 2 \sum_n (\varepsilon_n - E_n) \phi_n^2(x) = 0, \qquad (2)$$

where $\phi_n(x)$ is the nth normalized wave function. The functional minimization can be performed using a conjugate gradient method and the Numerov technique can be used to solve the discrete 1D SE. We have used this approach to successfully reproduce all of the numerical data for the Riemann zeros reported in Ref. [2]. Unfortunately, even on modern PC (Workstations), this direct approach involves a large computational undertaking. We have found it totally impractical to go beyond $N=2000$ levels when using this technique.

B. Method of the dressing transformation

The basis for the dressing transformation can be found in the soliton theory of nonlinear wave equations. The details of this approach are presented in Ref. [4], and we give here only the essential equations required for its implementation. The starting point is a given set of levels $\epsilon_0, \epsilon_1, \ldots, \epsilon_{N-1}, \epsilon_N$, which have been shifted so that one has exactly $N-1$ negative eigenvalues with the last eigenvalue satisfying $\epsilon_N = 0$. In this way, the potential is constructed from the "top down." This does not impose any restrictions other than the requirement that the number of levels to be fitted is fixed in advance. An initial potential $V(x)=0$ is then used as input and the first-order differential equation (diffeq)

$$f' - f^2 + V(x) = \epsilon \qquad (3)$$

is evaluated numerically with $f(0)=0$ and $\epsilon = \epsilon_{N-1}$. Once function f has been found at all points, a new potential $W(x)$ is constructed from the diffeq:

$$W(x) = 2\epsilon + 2f^2 - V(x). \qquad (4)$$

This new potential is then substituted back into Eq. (3) [i.e., $W(x) \rightarrow V(x)$] with the eigenvalue now being $\epsilon = \epsilon_{N-2}$. The choice $f(0)=0$ is enforced throughout the calculation and ensures that the final potential $V(x)$ is even. This iterative procedure is continued until all of the eigenvalues have been exhausted. The final $W(x)$ is the desired reconstructed potential that reproduces exactly all of the N eigenvalues. The integration of Eq. (3) can be performed using any standard method, but we have used a fourth-order Runge-Kutta (RK) method with step size $h = 1 \times 10^{-5}$ to ensure high accuracy. Nevertheless, it should be clear that the dressing transformation is numerically a far superior method since it does not require a large parameters' search in the construction of the potential. Consequently, it is easy to fit orders of magnitude more levels than in the direct SE approach.

C. Uniqueness of the reconstructed potentials

Having provided the basic background for numerically solving the inverse problem, we are now in a position to address the question of uniqueness. To this end, we first reconstruct the potential for the Riemann zeros with $N=500$. This number was chosen because it represents the maximum number of Riemann zeros fitted in Ref. [2]. Figure 1 displays the potential obtained from the direct SE approach (solid curve) and the dressing transformation (dashed). In the inset
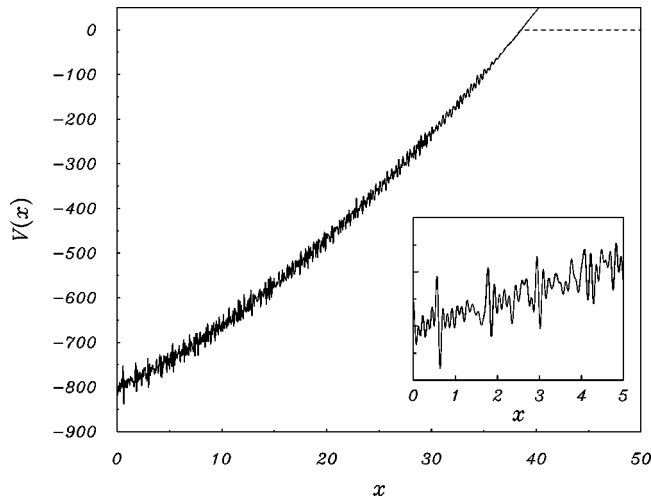
FIG. 1. The reconstructed potential for the Riemann zeros with $N = 500$ fitted levels. The dashed curve is obtained from the dressing transformation while the solid curve is from the direct SE approach. The inset shows an enlargement of the potential near the minimum. The two curves are indistinguishable below $x \approx 35$.

FIG. 2. As in Fig. 1, but for the prime numbers. Again, the curves are indistinguishable for energies below the last fitted level.

to the figure, we zoom in on the the region $x \in [0,5]$ to illustrate the remarkable agreement between the two methods, which continues until one approaches energies near the last fitted level ($x \approx 35$). Obviously, above this threshold the potentials cannot agree since the SE approach uses the semiclassical profile as its zeroth-order potential whereas the dressing transformation takes $V(x) = 0$. However, as the number of levels is increased, this threshold is moved to higher and higher energies, so that as $N \to \infty$, the two potentials should agree over all space. The equivalence between the two inversion techniques has been checked for up to $N = 2000$ (which computationally speaking, represents the upper limit of the SE approach), but we see absolutely no reason why this equivalence should not be true for arbitrarily large N. Nevertheless, to ensure that this agreement is not in some way fortuitous, we have also reconstructed the potential for the prime number sequence with $N = 500$. The results of the two approaches for the primes are shown in Fig. 2. Again, the potentials are indistinguishable except for regions near the last fitted level.

It also proves instructive to compare in more detail the reconstructed potentials for the Riemann zeros and prime numbers. To facilitate this comparison, we present in Fig. 3 enlargements of the potentials for $N = 2000, 5000, 10\,000$ levels around the minimum (one could look at other regions on the curves and obtain similar graphs). Figures 3(a)–3(c) correspond to the Riemann zeros whereas Figs. 3(d)–3(e) correspond to the prime potential. Several interesting observations can be made at this point. First, we note that for a given number of fitted levels, the Riemann zeros potential looks less "noisy" than the prime potential. In fact, by $N = 10\,000$, the prime potential takes on the characteristics of (roughly speaking) white noise, whereas the Riemann zeros appear to have far more structure. In particular, the Riemann potential contains large jumps at very specific spatial positions which appear to be quite robust (i.e., the positions of
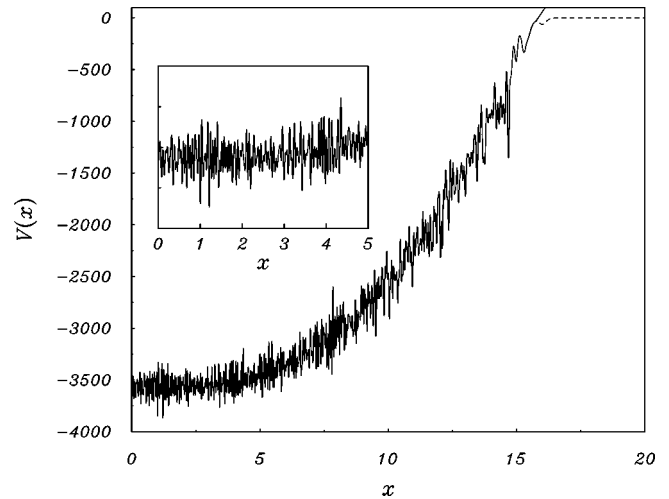
the jumps do not change as N increases). In contrast, any local structure present in the prime potential is completely washed out by $N = 10\,000$. This suggests that the prime potential may have a larger fractal dimension (i.e., close to $D = 2$ of white noise) than the Riemann zeros. Indeed, these observations lead us to consider more closely the fractal potentials generated by Ramani *et al.* [4].

As we mentioned earlier, the potentials in Ref. [4] were obtained from a generic GUE distribution without accounting for the Δ_3 statistics. A comparison of our Figs. 3(a)–3(c) with their Figs. 1(a)–1(c) clearly illustrates that a generic
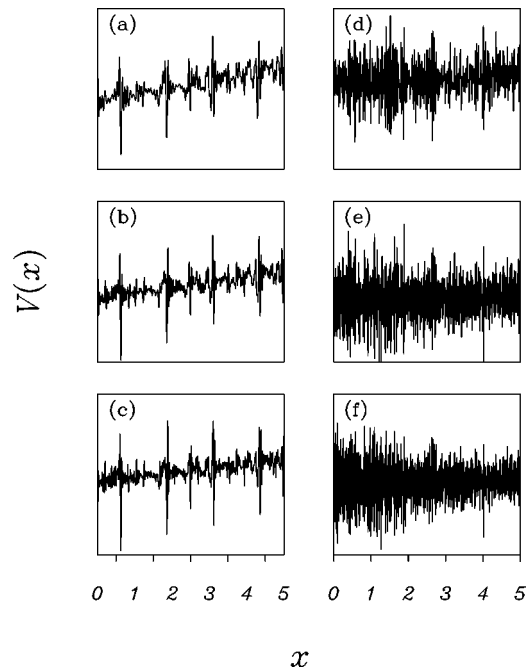

FIG. 3. Panels (a)–(c) and (d)–(f) correspond to the Riemann zeros and prime potential, respectively. The number of fitted levels from top to bottom is $N = 2000, 5000, 10\,000$. We have zoomed in on the range $x \in [0,5]$ to highlight the differences between the two potentials.

GUE distribution *cannot* capture, even qualitatively, the details of the Riemann zeros potential. We are therefore inclined to agree with the assertion in Ref. [6] that long-range correlations in the level spectrum play an important role in determining the fractal properties of the potential. This is further emphasized by noting that for large N, the prime potential [see Fig. 3(f)] looks very similar to the large N GUE fractal potential in Ref. [4], which was conjectured to have $D=2$ as $N\rightarrow\infty$. It is perhaps then not so ambitious to suggest that the prime potential also has fractal dimension $D=2$. We cannot, however, make such claims for the Riemann zeros, which evidently require a more careful quantitative analysis.

III. FRACTAL DIMENSION

We have seen in the preceding section that for the same number of fitted levels, the oscillatory behavior of the Riemann zeros and prime potentials are quite different. What significance then (if any) do these rapid oscillations have on the fractal dimension? Before we answer that question, it is perhaps useful to give a short account of the method used by the authors in Refs. [2,4] to estimate the fractal dimension of their reconstructed potentials.

The standard procedure for numerically estimating the fractal dimension of curves such as those we have generated is to use the box-counting technique [8]. Simply put, the box-counting technique involves choosing a region on the curve (over which the character of the oscillations does not change) and normalizing the axes so that the region is a square of side length equal to unity. The square region is then divided into n^2 cells of side length $\Delta x = 1/n$ and the number of cells P that contain a portion of the curve is counted. A plot of $\log(P\Delta x)$ versus $\log(\Delta x)$ is then constructed and the region over which the resulting graph has a constant slope $1-D$ determines the box dimension D; the box dimension is usually simply referred to as the fractal dimension. Although it is not the only method for estimating the fractal dimension of a curve, we will make use of the box-counting technique to allow for a sensible comparison with the results in Refs. [2,4]

In Fig. 4, we display the results of the box-counting method as applied to the Riemann potential for $N = 500, 5000, 10\,000$. In keeping with Ref. [2], we have focused on the range $x \in [0,10]$ for the analysis. What is clear from the figure is that the slope m of the linear region in the graph is essentially unchanged for $N > 500$. In fact, the slopes for $N = 5000$ and $N = 10\,000$ are virtually identical. Given that the fractal dimension is given by $D = 1 + |m|$, we conclude that to two significant figures, $D = 1.5$ for the the Riemann zeros potential. This result is consistent with the findings of Wu and Sprung [2] who likewise obtained $D = 1.5$. We see no evidence at all for $D = 2$ as was suggested by the work of Ramani *et al.* [4].

Figure 5 shows the results of the box-counting method for the prime number potential. In contrast to the Riemann zeros, there is a noticeable increase in the slope of the curve (i.e., larger fractal dimension) as one increases the number of fitted levels from $N = 500$ to $N = 5000$. However, for
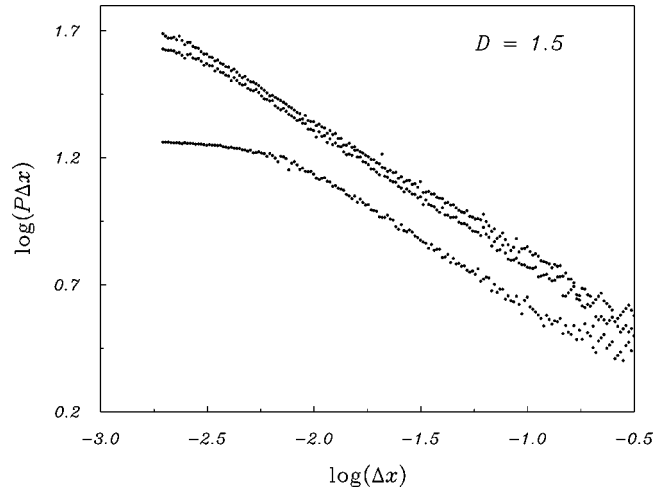

FIG. 4. Box-counting method analysis of the fractal dimension of the Riemann potential for (top to bottom) $N = 10\,000, 5000, 500$. The negative slope of the curve is the fractal dimension minus one.

$10\,000 < N < 40\,000$ (not shown), the slope is found to be practically unchanged. We are, therefore, led to conclude that the fractal dimension of the primes is $D = 1.8$. We are, however, open to the possibility that as $N \rightarrow \infty$ the fractal dimension of the prime potential approaches $D = 2$, based on the qualitative behavior of the potential seen in Figs. 3(d)–3(f). Indeed, a similar slow convergence of the fractal dimension was also observed in Ref. [4] for the GUE statistics. However, numerically investigating this convergence would require possibly levels of the order of 10^6, which is well beyond the capabilities of our current computational facilities.

IV. SUMMARY AND DISCUSSION

We have numerically examined the reconstruction of a 1D local potential from a set of energy levels using two distinct
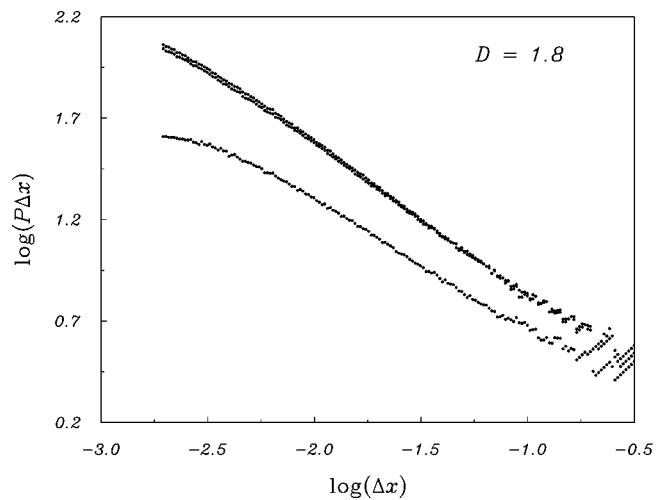

FIG. 5. Box-counting method analysis of the fractal dimension of the prime potential with (top to bottom) $N = 10\,000, 5000, 500$. The negative slope of the curve is the fractal dimension minus one. Note that the difference in the curves from $N = 5000$ to $N = 10\,000$ is negligible.

inversion procedures. Applying these methods to the Riemann zeros and the prime number sequence we find that both inversion techniques produce the same quantum potential when applied to the same set of energy levels. This provides evidence that the inversion problem in 1D results in a unique potential. Thus, the answer to the first question raised in the Introduction is affirmative. Based on this finding, it follows that the fractal dimension cannot depend on the method of inversion. We have also examined the issue of the fractal dimension of the Riemann zeros for up to $N = 40\,000$ levels and estimate that $D = 1.5$. This result is in complete agreement with Wu and Sprung's [2] earlier estimate (albeit using only $N = 500$ levels), and illustrates that the potentials investigated by Ramani *et al.* [4] have little bearing on the Riemann zeros potential. In particular, we demonstrated that it is insufficient to simply account for the NNSD of the energy levels to capture the local details responsible for the fractal properties of the reconstructed potential; the Riemann zeros display local structure that is not reproduced by a generic GUE level spectrum. Therefore, long-range correlations appear to play a pivotal role in determining the fractal dimension of the potential.

The prime number potential has also been investigated and found to have fractal dimension $D = 1.8$. This result is somewhat surprising since the NNSD of the primes is almost poissonlike, whereas the Riemann zeros obey the (chaotic) GUE statistics. Unlike the Riemann potential, our calculation of the prime potential's fractal dimension suggests that there may be a very slow convergence (as $N \to \infty$) of the fractal dimension to $D = 2$. This possibility has also been noted in Ref. [4] where the fractal properties of white-noise-like potentials corresponding to the GUE statistics were studied.

ACKNOWLEDGMENTS

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